Let $B^n(r)$ denote the closed ball of radius r in $\mathbb{R}^n$ centered at the origin.

(a) Show that $\mathrm{Vol}(B^n(r)) = \lambda_n r^n$ for some positive constant λ_n.

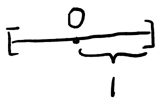
(b) Compute λ_1 and λ_2.

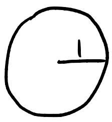
(c) Compute λ_n in terms of λ_{n-2}.

(d) Deduce a formula for λ_n for general n. *(Hint: consider two cases, according to whether n is even or odd.)*

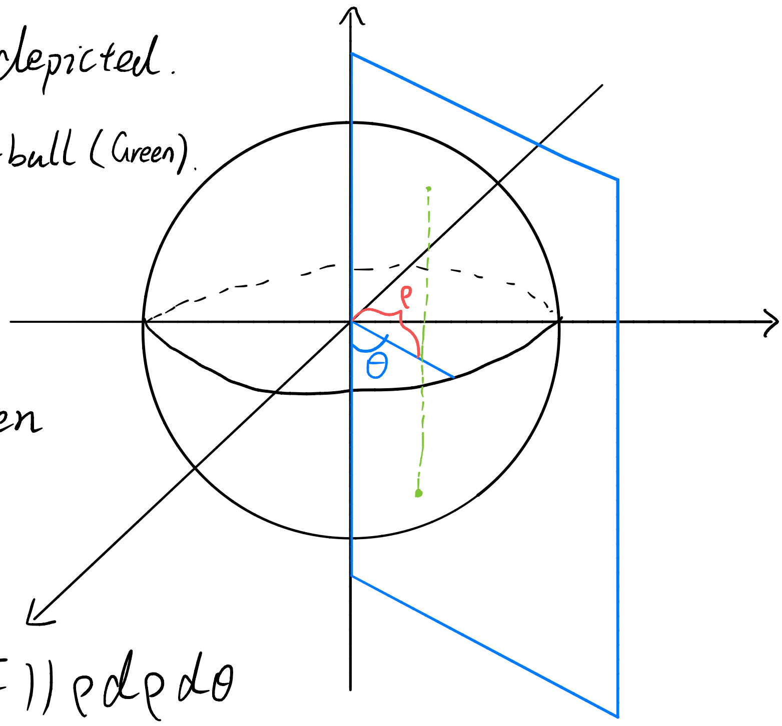
(a) $\mathrm{Vol}(B^n(r)) = \int_{B^n(r)} dx_1 \cdots dx_n \overset{\substack{f \\ \downarrow \\ y_n = r x_n}}{=\!=} \int_{B^n(1)} |\det Df| \, dy_1 \cdots dy_n$

$= \int_{B^n(1)} r^n \, dy_1 \cdots dy_n = \underbrace{\mathrm{Vol}(B^n(1))}_{\lambda_n} r^n$

(b) $\lambda_1 = \mathrm{Vol}(B^1(1)) = \mathrm{Vol}($  $) = 2$

$\lambda_2 = \mathrm{Vol}(B^2(1)) = \mathrm{Vol}($  $) = \pi$

(c) Let ρ and θ be defined as depicted.
For fixed ρ and θ, they define a $(n-2)$-ball (Green)
of radius $\sqrt{r^2-\rho^2}$.



We can slice $B^n(r)$ by these green $(n-2)$-balls. Thus

$$\mathrm{Vol}(B^n(r)) = \int_0^{2\pi}\int_0^r \mathrm{Vol}(B^{n-2}(\sqrt{r^2-\rho^2}))\,\rho\, d\rho\, d\theta$$

$$\overset{(a)}{=} 2\pi \mathrm{Vol}(B^{n-2}(r)) \int_0^r \left(1-\left(\frac{\rho}{r}\right)^2\right)^{\frac{n-2}{2}} \rho\, d\rho$$

$$= \frac{2\pi r^2}{n} \mathrm{Vol}(B^{n-2}(r))$$

(Also, can be computed by using high dimensional spherical coor).

(d) $\lambda_{2k} = \dfrac{\pi^k}{k!}$, $\lambda_{2k+1} = \dfrac{2k!(4\pi)^k}{(2k+1)!}$

Remark: As $n \to \infty$, $\lambda_n \to 0$, $Vol(B^\infty(1)) = 0$

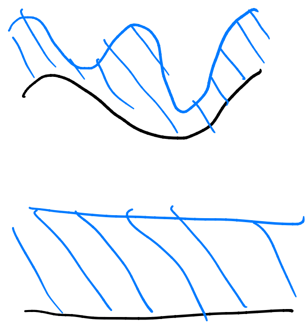
$\overline{B^\infty(1)} \subset \mathbb{R}^\infty$ is closed, bounded, but not compact.

Recall:

How to Evaluate a Line Integral

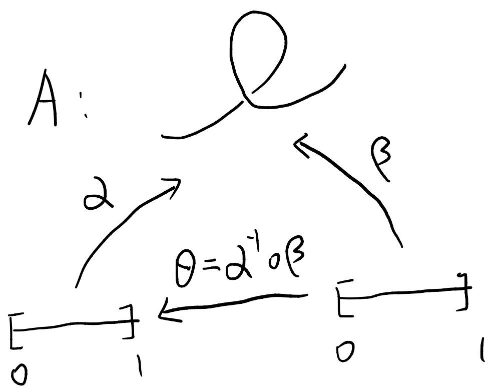
To integrate a continuous function $f(x, y, z)$ over a curve C:

1. Find a smooth parametrization of C,
$$\mathbf{r}(t) = g(t)\mathbf{i} + h(t)\mathbf{j} + k(t)\mathbf{k}, \qquad a \le t \le b.$$
2. Evaluate the integral as
$$\int_C f(x, y, z)\, ds = \int_a^b f(g(t), h(t), k(t))|\mathbf{v}(t)|\, dt.$$



Q: Why is the integral independent of the parametrizations?

A:



Let α, β be two parametrizations $[0,1] \to \mathbb{R}^n$.

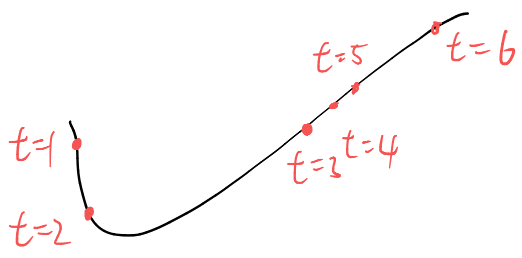
$\exists\ \theta : [0,1] \to [0,1]$ s.t. $\beta = \alpha \circ \theta$. Then

$$\int_0^1 f(\beta(t)) \cdot \beta'(t)\, dt = \int_0^1 f(\alpha(\theta(t))) \cdot \alpha'(\theta(t)) \cdot \theta'(t)\, dt$$

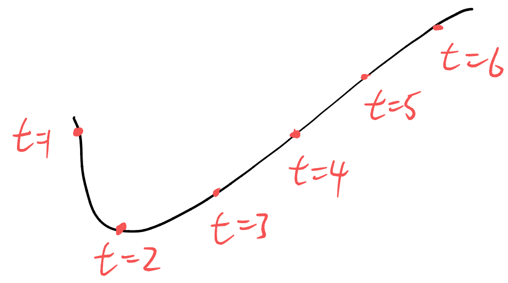
$$\overset{u=\theta(t)}{=} \int_0^1 f(\alpha(u)) \cdot \alpha'(u)\, du$$

Q: Among all the parameterizations, is there the most natural one?

A: parameterization by arc length:



not by arc length



by arc length

Def $\alpha : [a,b] \to \mathbb{R}^n$ is called a parameterization by arc length if $|\alpha'(t)| \equiv 1$.

In this case the arc length can be computed by $\int_c^d dt = d-c$

Q: How to find the parameterization by arc length?

A: Suppose $\beta : I \to \mathbb{R}^n$ is a parameterization

The arc length is $s(t) = \int_a^t |\beta'| \, dt$.

Now solve for t as $t = t(s)$ to get $t : J \to I$

Let $\alpha(s) = \beta \circ t$. Then

$$|\alpha'(s)| = \left| \frac{d\beta}{dt} \cdot \frac{dt}{ds} \right| = \left| \beta'(t) \cdot \frac{1}{|\beta'(t)|} \right| \equiv 1$$